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Quantum control and representation theory

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Abstract
A new notion of controllability for quantum systems that takes advantage of the linear superposition of quantum states is introduced. We call such a notion von Neumann controllability, and it is shown that it is strictly weaker than the usual notion of pure state and operator controllability. We provide a simple and effective characterization of it by using tools from the theory of unitary representations of Lie groups. In this sense, we are able to approach the problem of control of quantum states from a new perspective, that of the theory of unitary representations of Lie groups. A few examples of physical interest and the particular instances of compact and nilpotent dynamical Lie groups are discussed.

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1. Introduction: control of infinite-dimensional quantum systems

The theory of control of quantum systems has had a strong influence from ideas from the classical theory of control. In fact, from a purely mathematical perspective, a quantum system is nothing but a linear evolution equation on a vector space; thus the methods and ideas in classical control apply straightforwardly. This approach has been very successful in tackling control problems for finite-dimensional quantum systems (or finite-dimensional approximations to them), where well-known theorems characterizing the various notions of controllability of classical systems have been applied. For instance, the state controllability of a finite-dimensional quantum system has been considered from a Lie-theoretical perspective determining necessary and sufficient conditions for a given dynamical group to act transitively on the set of normalized pure states (see, for instance, the recent book [1] and references therein). However, such ideas cannot be straightforwardly extended to the infinite-dimensional situation for various reasons, among them the intrinsic and unavoidable analytical difficulties coming from the appearance of unbounded operators and the complications inherent to infinite-dimensional geometry.

Moreover, the results obtained for finite-dimensional approximations of quantum systems do not extend naively to the infinite-dimensional case. For instance, it can be easily checked that the harmonic oscillator control problem,

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} + \left(\frac{1}{2}q^2 - u(t)q\right)\psi, \tag{1.1}$$

is not controllable (see, for instance, [13]); however, its truncations up to the first $n + 1$ eigenstates, energies between $1/2$ and $n + 1/2$, whose form is

$$\mathrm{i}\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t} = (\tilde{H}_0 + u\tilde{H}_1)\tilde{\psi}, \qquad \tilde{\psi} \in \mathbb{C}^{n+1},$$

are controllable for every n [15]. A few results on the controllability of the 1D Schrödinger equation have appeared recently [2, 3] displaying some of the subtleties of the infinite-dimensional case. It is important to point out that a rigorous approximate controllability theorem has been proved for general bilinear systems under a non-rational resonance condition of the spectrum of the free Hamiltonian by Chambrion *et al* [6].

Other recent contributions to the subject have been focusing on the possibility of circumventing the technical difficulties that arise in the infinite-dimensional setting. For instance, Bloch *et al* [4] provide a set of sufficient conditions to prove the controllability of finite-dimensional systems coupled to harmonic oscillators that extend the well-known conditions in finite-dimensional geometry. There are also remarkable results by Clark [7] on controllability based on the existence of a common domain of analytic vectors for the control Hamiltonians H_k.

In this work, we take a different approach to the problem of state controllability of quantum systems by relaxing the notion of (approximate) control and allowing for the linear superposition of states. In fact, the linear superposition principle constitutes a fundamental ingredient of quantum physics; therefore, it can be exploited when addressing the problem of control of quantum systems, in contrast to what happens in classical systems. Moreover, the use of the superposition principle has attracted a lot of attention since the early years of quantum physics and is becoming a more and more relevant tool in the manipulation of quantum states. Sometimes superposed states are called cat states recalling the famous thought experiment by Schrödinger. Linear superposition of states has been achieved, for both photons and electrons, for various purposes (see, for instance, [8, 12] for recent applications to quantum information theory). We refer in this paper to a few concrete experiments with photons, in particular a Mach–Zehnder interferometer with a Kerr medium, that will help us to illustrate this new notion of controllability [9, 11].

Thus, given a control quantum Hamiltonian $H(u)$, we ask for the existence of controls such that the target state gets arbitrarily close to a linear superposition of states, each one obtained evolving from a given initial one by the use of (possibly) different families of control functions. We call such a notion of controllability von Neumann controllability for reasons that will become obvious from the discussion to follow, and we will relate this notion of state control to the theory of unitary representations of the dynamical Lie group of the system. In fact, a necessary condition for the pure state controllability of the system is the irreducibility of the corresponding unitary representation of the dynamical group. It is also shown that the irreducibility of the unitary representation of the dynamical Lie group is equivalent to von Neumann controllability of the system. Thus if the dynamical group G is compact, we are led necessarily to consider finite-dimensional representations. Consequently, genuine state controllable infinite-dimensional systems can only arise if the group G is not compact. Some families of infinite-dimensional representations are well known, for instance, for nilpotent and solvable groups. We use the knowledge we have on such representations to obtain some simple results on the various notions of controllability for such systems.

The plan of this paper is as follows. Section 2 is devoted to establishing the basic notions of controllability and approximate controllability of quantum systems and to looking into the differences between the finite- and infinite-dimensional cases. We introduce the notion of von Neumann controllability in section 3 and discuss its relation to the representation theory of the dynamical Lie group of the system in section 4. Some relations between the various notions of controllability are discussed there and an explicit characterization of von Neumann controllability is given. Finally, a discussion takes place in section 5 about various, particularly interesting, cases related to compact and nilpotent groups, as well as the oscillator algebra and its relation to the Mach–Zehnder–Kerr system discussed in section 3, followed by a summary of the paper and an outlook on further developments.

2. Controllability and approximate controllability for quantum systems

We shall consider a quantum system defined on a Hilbert space $\mathcal{H}$ with a Hamiltonian operator $H(u)$ that depends on a family of control functions $u(t)$ lying in some class $\mathcal{U}$ of admissible controls. Typically, the control Hamiltonian will be of affine-linear type, i.e.

$$H(u) = H_0 + \sum_{k=1}^{r} u_k(t) H_k,$$

where the drift Hamiltonian H_0 represents the dynamics of the uncontrolled (or 'free') system, and the set of controls $\mathcal{U}$ is the space of bounded piecewise constant functions $u(t)$. Given a family of control functions $u(t)$, if $H(u(t))$ is a self-adjoint operator for t in the interval $[0, T]$, there exists a unique solution to the time-dependent Schrödinger equation:

$$\mathrm{i}\hbar\dot{\psi} = H(u(t))\psi, \tag{2.1}$$

with the initial state $\psi_0 \in \mathcal{H}$. We shall denote such a solution by $\psi(t; u)$. Moreover, there exists a family of unitary operators $U(t)$ which provide the quantum evolution of the system on Heisenberg's representation, i.e.

$$\mathrm{i}\hbar\dot{U} = H(u(t))U, \qquad U(0) = I, \qquad U \in U(\mathcal{H}),$$

where $U(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$.

For a given time $t > 0$ we define the set of reachable states (or operators) $\mathcal{R}_t(\psi_0) = \{\psi = \psi(t; u) \in \mathcal{H} | u \in \mathcal{U}\}$ $\left(\mathcal{R}_t^{\mathrm{op}} = \{U = U(t; u) \in U(\mathcal{H}) | u \in \mathcal{U}\}\right)$, and for a given $T > 0$ (T could be $+\infty$) we consider the set of reachable states (or operators) for all times $0 \leqslant t \leqslant T$, that is $\mathcal{R}(\psi_0, T) = \bigcup_{0\leqslant t\leqslant T} \mathcal{R}_t(\psi_0)$ $\left(\mathcal{R}^{\mathrm{op}}(T) = \bigcup_{0\leqslant t\leqslant T} \mathcal{R}_t^{\mathrm{op}}\right)$. Then, given a subset $\Sigma \subset \mathcal{H}$ and a time $T > 0$, we say that the system is (Σ, T)-controllable with respect to the state ψ_0 if $\Sigma \subset \mathcal{R}(\psi_0, T)$, and we say that the system is (Σ, T)-controllable if it is (Σ, T)-controllable with respect to any ψ_0. Typically, we will be interested in pure state controllability; this is $\Sigma = S \subset \mathcal{H}$, where S is the unit sphere made up of all unitary vectors on the Hilbert space $\mathcal{H}$ and $T = +\infty$. Similarly, we say that the system is operator controllable if $\mathcal{R}^{\mathrm{op}}(+\infty) = U(\mathcal{H})$.

There are various criteria for assessing the problem of controllability in finite-dimensional systems; however, not much is known in the infinite-dimensional situation. The most successful methods in finite dimension for affine quantum control systems are based on the identification of the reachable set with an orbit of the Lie group associated with the dynamical Lie algebra of the system, i.e. the unique connected and simply connected Lie group G such that its Lie algebra $\mathfrak{g}$ is the Lie algebra generated by the operators $\mathrm{i}H_0, \ldots, \mathrm{i}H_r$. Hence an n-dimensional affine quantum control system is pure state controllable if the Lie group G acts transitively on the finite-dimensional sphere S^{2n-1}. It is then a simple task based on classical results on homogeneous spaces by Montgomery [14] to characterize those dynamical Lie algebras that lead to controllability as was done by Brockett in the classical theory of control [5] and more recently in the quantum case [17, 18].

The geometrical ideas that lie behind the techniques to study controllability in finite-dimensional systems can hardly be extended to the infinite-dimensional case. The knowledge we have on infinite-dimensional groups and their realizations is considerably less than we have on finite-dimensional ones. In particular, we do not have theorems characterizing groups possessing spheres of infinite dimension as homogeneous spaces like in the finite-dimensional case. Moreover, more fundamental difficulties arise in infinite-dimensional Hilbert spaces from the fact that in most occasions the operators $H_0, \ldots, H_r$ are unbounded, and the construction of the dynamical Lie algebra is compromised. As was commented in the introduction, various partial results are known that use either strong restrictions on the domains and the structure of the Lie algebras generated by the operators H_k as in [7], or in the structure of the Hilbert space $\mathcal{H}$ as in [4]. In both cases, criteria for the controllability of some interesting systems are obtained.

Inspired by practical considerations, a weaker notion of controllability known as approximate controllability is also used. Given an initial state ψ_0, we say that a quantum control system $H(u)$ is ϵ-approximately controllable with respect to ψ_0, if for any given state ψ, there exists a time T and a control function $u(t) \in \mathcal{U}$ such that the solution $\psi(t; u)$ satisfies $\|\psi(T; u) - \psi\| < \epsilon$. If the system is ϵ-approximately controllable for all $\epsilon > 0$ and for all ψ_0 we say that it is approximately controllable. The notion of approximate controllability is suitable for experimental purposes where absolute precision is impossible to achieve. Note that a quantum system is approximately controllable if the reachable set $\mathcal{R}$ is dense in S. In this sense, it is notable the result recently obtained by Chambrion *et al* [6], where it is proven that a bilinear system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = (A + uB)\psi, \qquad \psi \in \mathcal{H},$$

where A, B are skew-adjoint operators with discrete spectrum, such that the sequence of differences of eigenvalues of $A, \lambda_n - \lambda_{n+1}$, is rationally independent, the elements $\langle \phi_n, B\phi_{n+1}\rangle \neq 0$ and u is a piecewise constant function, is approximately controllable. In particular, if we write the harmonic oscillator equation (1.1) in the form $\mathrm{i}\dot{\psi} = (H_0 + uH_1)\psi$ and now consider the bilinear system $\dot{\psi} = (A + uB)\psi$ with $A = -\mathrm{i}(H_0 + \mu H_1)$ and $B = -\mathrm{i}(H_1 - \mu H_0)$ for μ an irrational real number small enough, then the conditions of the previous theorem are fulfilled and the system is approximately controllable.

3. Von Neumann controllability

3.1. A simple linear superposition quantum control device based on a Mach–Zehnder–Kerr interferometer

As was discussed in the introduction, there are a number of possibilities for generating quantum superposition states, and hence to try to implement a device that would allow us to control states by using linear superposition. Before embarking on the formal definition of von Neumann controllability we will discuss first one of these methods, which is based on the generation of optical macroscopic superposition states via state reduction using a Mach–Zehnder interferometer with a Kerr medium as was discussed in [9]. We will see that, in addition, this experiment provides effective control mechanisms for the output quantum states. A schematic for the method is given in figure 1. The device requires a standard

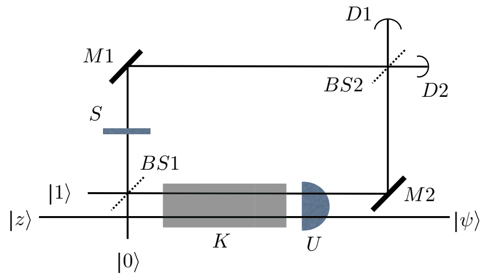

Figure 1. Scheme for the proposed method to construct linearly superposed states. (This figure is in colour only in the electronic version)

Mach–Zehnder interferometer with a Kerr medium (K) in one arm of the counterclockwise path and a phase shifter (S) in the clockwise path generating a phase shift θ.

The quantum state to be controlled is a coherent state $|z\rangle$, which is a vector on the Fock–Hilbert space describing the quantum states of optical laser photons. We would like to obtain output states which are linear superpositions of the form $|\psi\rangle = (c_1 I + c_2 U_2 + \cdots + c_N U_N)|z\rangle$, where c_k are complex numbers, and U_k are unitary operators, $k = 1, \ldots, N$.

Denoting by $\hat{a}, \hat{a}^\dagger$ the annihilation and creation operators on the Fock space of the laser, $|z\rangle$ is an eigenvector of $\hat{a}$ with the eigenvalue z, $\hat{a}|z\rangle = z|z\rangle$. To begin describing the formation of Schrödinger cat states we assume that a single photon and vacuum states $|1\rangle$, $|0\rangle$ enter the input ports of the first beam splitter (BS1) of the interferometer, as indicated in figure 1, and that the coherent state $|z\rangle$ enters the channel that goes through the Kerr medium. Just after the first beam splitter the state of the system is $\frac{1}{\sqrt{2}}(|T\rangle + \mathrm{i}|R\rangle)\,|z\rangle$, where $|R\rangle$ and $|T\rangle$ denote, respectively, the states in which the photon has been reflected and transmitted. The interferometer states $|R\rangle, |T\rangle$ are entangled, consisting of a superposition of the states for a photon propagating along two different paths, the two arms of the interferometer. Just before the mirrors (M1) and (M2), as a result of the Kerr interaction and the phase shifter, the state of the system is

$$\frac{1}{\sqrt{2}}(\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}|R\rangle|z\rangle + |T\rangle|z\,\mathrm{e}^{-\mathrm{i}\phi}\rangle), \tag{3.1}$$

where $\phi = Kl/v$ has time units, K measures the strength of the Kerr medium, l is its length, and v is the velocity of the light in the medium. Now, at the mirrors (M1) and (M2) both beams suffer a $\pi/2$ phase shift amounting to an overall irrelevant phase factor. The second beam splitter, according to the superposition principle, performs the following transformations:

$$|R\rangle = \frac{1}{\sqrt{2}}(|D_2\rangle + \mathrm{i}|D_1\rangle), \qquad |T\rangle = \frac{1}{\sqrt{2}}(|D_1\rangle + \mathrm{i}|D_2\rangle)$$

and the state of the system given by (3.1) finally becomes

$$\tfrac{1}{2}[|D_1\rangle(|z\,\mathrm{e}^{-\mathrm{i}\phi}\rangle - \mathrm{e}^{\mathrm{i}\theta}|z\rangle) + \mathrm{i}|D_2\rangle(\mathrm{e}^{\mathrm{i}\theta}|z\rangle + |z\,\mathrm{e}^{-\mathrm{i}\phi}\rangle)]. \tag{3.2}$$

If detector D1 and not D2 fires, then the state $|D_1\rangle$ is detected, thus projecting the total state (3.2) into the state on the Fock space of the laser given by $|\psi\rangle = |z\,\mathrm{e}^{-\mathrm{i}\phi}\rangle - \mathrm{e}^{\mathrm{i}\theta}|z\rangle$. A simple computation shows that the probability of obtaining this state (i.e. the probability that the state $|D_1\rangle$ is detected) is given by $P(\theta,\phi) = \frac{1}{2}\{1 - \exp[-|z|^2(1-\cos\phi)]\cos(\theta + |z|^2\sin\theta)\}$. Now, the state $|z\,\mathrm{e}^{-\mathrm{i}\phi}\rangle$ is obtained as a unitary evolution of state $|z\rangle$ with respect to the harmonic oscillator Hamiltonian $H_0 = \hat{a}^\dagger\hat{a} + 1/2$ as $\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi H_0}|z\rangle = |z\mathrm{e}^{-\mathrm{i}\phi}\rangle$. Denoting the unitary operator $\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi H_0}$ by U_ϕ we get that the output state could be written as $|\psi\rangle = (\mathrm{e}^{\mathrm{i}\theta} I + U_\phi)|z\rangle$ which has the form that we were looking for.

Moreover, if we insert a usual quantum gate U between the Kerr medium and the mirror (M2), see figure 1, coupling the interferometer and the laser channel, of the form,

$$U(|R\rangle|z\rangle) = \cos\alpha|R\rangle U_1|z\rangle + \sin\alpha|T\rangle U_2|z\rangle,$$
$$U(|T\rangle|z\rangle) = -\sin\alpha|R\rangle U_1|z\rangle + \cos\alpha|T\rangle U_2|z\rangle,$$

where U_k, $k = 1, 2$ are unitary operators on the Fock space, we get that the final output state $|\psi\rangle$ above becomes

$$|\psi\rangle = (\mathrm{e}^{\mathrm{i}\theta}I + \mathrm{i}\cos\alpha U_1 U_\phi - \sin\alpha U_2 U_\phi)|z\rangle. \tag{3.3}$$

Now the parameter ϕ, which is proportional to the length of the Kerr medium, and the pase shift θ act as control parameters that together with the quantum gates U_k allow us to generate a superposition of quantum states each one evolved by a different unitary operator.

3.2. *On the general notion of linear superposition quantum controllability*

We assume now that we have an affine-linear quantum control system $H(u) = H_0 + \sum_{k=1}^{r} u_k H_k$ such that the family of self-adjoint operators $H_0, \ldots, H_r$ generates a finite-dimensional Lie algebra $\mathfrak{g}$ (i.e. the skew symmetric operators $\mathrm{i}H_k$ are the generators of the Lie algebra $\mathfrak{g}$). In such a case, the dynamical Lie group G corresponding to the dynamical Lie algebra $\mathfrak{g}$ is represented unitarily on the Hilbert space $\mathcal{H}$ of the system. If we denote by A_l a basis of the Lie algebra $\mathfrak{g}$, then any element g on the group G can be written as

$$g = \prod_{r<\infty} \exp \tau_{l_r} A_{l_r}, \tag{3.4}$$

for some family of real numbers τ_{l_r}. The solutions of Schrödinger's equation (2.1) for piecewise constant control functions $u_k(t)$ consist of piecewise differential curves $\psi(t; u)$ of the form,

$$\psi(t; u) = U(t, t_s)U(t_s, t_{s-1}) \cdots U(t_2, t_1)U(t_1, t_0)\psi_0,$$

where $u(t)$ is defined on the interval $[t_0, T]$ with discontinuity points $t_1 < t_2 < \cdots < t_f$, $t_s \leqslant t < t_{s+1}$, and $U(t_j, t_{j-1})$ are unitary operators of the form,

$$U(t', t) = \prod_{j<\infty} \mathrm{e}^{\tau_{l_j}\hat{A}_{l_j}}, \tag{3.5}$$

with $\hat{A}_{l_j}$ being the skew-Hermitian operators realizing the basis elements A_{l_j} (note that the operators $\hat{A}_l$ realizing the basis A_l are linear combinations of commutators of various orders of the elements $\mathrm{i}H_k$). Mimicking equation (3.3) for families of unitary operators of the form (3.5), we propose the following natural notion of controllability.

Definition 1. *An affine quantum control system with a finite-dimensional dynamical Lie algebra $\mathfrak{g}$, will be said to be von Neumann controllable with respect to the state ψ_0 if for any state ψ_1 and any $\epsilon > 0$, there exists a family of coefficients c_k, $k = 1, \ldots, N$ (N depending on ϵ), and for each k a family of times t_{k_l}, $l = 1, \ldots, M_k$, such that*

$$\left\| \psi_1 - \sum_{k=1}^{N} c_k \prod_{l=1}^{M_k} e^{t_{k_l}\hat{A}_{k_l}} \psi_0 \right\| < \epsilon,$$

where A_j is a basis of the dynamical Lie algebra of the system. Moreover, we will say that the system is von Neumann controllable if it is von Neumann controllable with respect to any state ψ_0.

The notion of von Neumann controllability is a notion of approximate controllability. It is clear from the definitions and the discussion above that an approximately controllable quantum system is von Neumann controllable. However, the contrary is not necessarily true as will be discussed later on.

Let us insist that in sharp contrast with the notion of pure state controllability, von Neumann controllability allows for the use of linear superposition of states to reach the target states. The algebra of operators on a Hilbert space generated by operators of the form (3.5) is called a von Neumann algebra and this is the reason for the name chosen for the above notion of controllability.

4. Von Neumann controllability and representation theory

It is now time to discuss the relation of von Neumann controllability with the unitary representations of the dynamical group of the system.

We should note that for each element g of the dynamical group of the form (3.4), we have a unitary operator associated

$$U(g) = \prod_{s<\infty} \mathrm{e}^{t_{l_s}\hat{A}_{l_s}}. \tag{4.1}$$

Thus the map $U(g)$ provides a unitary representation of the group G. In the particular case that the operators $\mathrm{i}H_k$ do define a basis for a Lie algebra, i.e. they are independent and satisfy commutation relations of the form

$$[H_k, H_j] = \mathrm{i}C^l_{kj}H_l, \qquad k.j, l = 0, \ldots, r,$$

the group G is represented by the unitary operators $U_k(t) = \mathrm{e}^{\mathrm{i}tH_k}$. It is then clear that the unitary representation $U: G \to U(\mathcal{H})$ is strongly continuous because for any vector $\psi \in \mathcal{H}$ the map $G \to \mathcal{H}$ given by $g \mapsto U(g)\psi$ is continuous.

Thus in the study of controllability of affine quantum systems we are led naturally to consider its relation with the theory of unitary representations of groups. In fact, the first simple observation in this sense is the following.

Theorem 1. *Let us consider an affine-linear quantum control system $H(u) = H_0 + \sum_{k=1}^r u_k H_k$ such that its dynamical group G is a finite-dimensional Lie group. Then a necessary condition for the pure state controllability of the system is that the unitary representation U of the dynamical group is irreducible.*

Proof. Denoting by $U(g)$ the unitary representation of the dynamical Lie group G on $\mathcal{H}$, it is clear that if U is not irreducible, then there exists a proper invariant closed subspace $W \subset \mathcal{H}$. Note that the unitary operators $U(g)$ leave invariant the pure states of the system $U(g)S \subset S$, for all $g \in G$. Therefore, it is clear that $W \cap S$ will be an invariant subset of S strictly contained on it. Hence, there will exist vectors $\psi \in S$ such that $\psi \notin W \cap S$. Such vectors will not be reachable from any vector on W. □

Given a unitary representation U of a Lie group G on a Hilbert space $\mathcal{H}$, it is natural to consider for any vector $\psi \in \mathcal{H}$ its 'linear orbit'; this is the linear closure of all vectors of the form $U(g)\psi$, $g \in G$, or in other words the linear closure of the actual orbit of the vector ψ under the action of the group G. Such a linear subspace will be denoted by $\mathcal{H}^U_\psi$, i.e.

$$\mathcal{H}^U_\psi = \overline{\mathrm{span}\{U(g)\psi | g \in G\}}.$$

A vector ψ is called cyclic if $\mathcal{H}^U_\psi = \mathcal{H}$ and the corresponding representation is said to be cyclic. Hence it is clear that the quantum system of theorem 1 will be von Neumann controllable with respect to the state ψ_0 if ψ_0 is a cyclic vector for the unitary representation U.

It is also clear that any nonzero vector on the vector space of an irreducible representation U is cyclic. Conversely, if all vectors of an unitary representation are cyclic, the representation is irreducible. Taking advantage of this we can state the following theorem that provide us with a sharp criterion to determine when an affine quantum system is going to be von Neumann controllable.

Theorem 2. *An affine-linear quantum system* $H(u) = H_0 + \sum_{k=1}^{r} u_k H_k$ *with finite-dimensional dynamical Lie algebra* $\mathfrak{g}$ *is von Neumann controllable if and only if the unitary representation* U *of its dynamical Lie group* G *is irreducible.*

Proof. If the unitary representation $U(g)$ is irreducible, every nonzero vector ψ_0 is cyclic. Then because $\mathcal{H} = \overline{\text{span}\{U(g)\psi | g \in G\}}$, the set of finite linear combinations of vectors $U(g)\psi$ is dense on $\mathcal{H}$. Hence for any vector ψ_1 and for any $\epsilon > 0$ there exists a family of elements g_i and constants c_i, $i = 1, \ldots, m$, such that

$$\left\| \psi_1 - \sum_{i=1}^{m} c_i U(g_i) \psi_0 \right\| < \epsilon.$$

Moreover, any unitary operator of the representation $U(g)$ can be written as in equation (4.1). Substituting back in the previous formula we find the expression defining von Neumann controllability.

Conversely, if the quantum system is von Neumann controllable, then for any nonzero vector ψ we have that $\mathcal{H} = \overline{\text{span}\{U(g)\psi | g \in G\}}$; thus any nonzero vector is a cyclic vector for the unitary representation $U(g)$. Hence the representation is irreducible. □

However, we should stress that reducible representations of groups can contain cyclic vectors. This opens the possibility of von Neumann control with respect to a given vector even if the unitary representation of the group is reducible, obtaining in this way controllability with respect to a given vector by using just a few controls and providing a way of improving the efficiency of the control methods. For instance, if we consider a spin $1/2$ system, it trivially supports an irreducible unitary representation of the group $SU(2)$; however, such a representation is reducible for any $U(1)$ subgroup of $SU(2)$. Now if we select a $U(1)$ subgroup of $SU(2)$ and consider the north–south axis defined by it on S^2, then any non-equatorial vector is a cyclic vector with respect to the (reducible) unitary representation defined by the given subgroup $U(1)$ on $\mathbb{C}^2$. This system is then von Neumann controllable for a wide subspace of its Hilbert space but the control group is much smaller than that needed to achieve pure state controllability, i.e., $SU(2)$. Of course the question of whether a given initial state is cyclic or not in a reducible representation is still open and we leave this issue for future work.

5. Some examples and applications

5.1. Von Neumann control of coherent states on a Mach–Zehnder–Kerr interferometer

We are ready now to finish the discussion of the controllability of coherent states on a Mach–Zehnder–Kerr interferometer started in section 3. In order to do that let us consider again the example of the controlled quantum harmonic oscillator (1.1). The controlled harmonic oscillator has a dynamical algebra generated by the self-adjoint operators

$$H_0 = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2, \qquad H_1 = -q,$$

on the Hilbert space $L^2(\mathbb{R})$. Because of the commutation relations,

$$[H_0, H_1] = \mathrm{i}H_2, \qquad [H_0, H_2] = -\mathrm{i}H_1, \qquad [H_1, H_2] = -\mathrm{i}I,$$

with $H_2 = \mathrm{i}\frac{\partial}{\partial q}$, the dynamical Lie algebra of this system is a four-dimensional Lie algebra called the oscillator algebra and the Lie group defined by it the oscillator group. Such a group is a four-dimensional solvable Lie group whose unitary representations were thoroughly studied for the first time in a remarkable paper by Streater where a detailed comparative analysis of the construction of its unitary irreducible representations using both Mackey's theory and Kirillov's coadjoint orbit quantization construction was done [19]. Hence, according to Streater's classification the previous representation of the oscillator group is irreducible (in fact, it is essentially the only one that has physical sense) and the system is von Neumann controllable. However, we already know that it is not pure state controllable. This shows that the notion of von Neumann controllability is different and strictly weaker than the notion of pure state controllability. We also know that the controlled quantum harmonic oscillator is approximately controllable. Thus, in this case, von Neumann controllability and approximate pure state controllability are equivalent.

We can apply this result to the example of a Mach–Zehnder–Kerr interferometer if in addition to the harmonic oscillator Hamiltonian $H_0 = \hat{a}^\dagger\hat{a} + 1/2$ that describes the interaction of the Kerr medium with the coherent state, we introduce another medium whose effective Hamiltonian has a term $H_1 = -q = -(\hat{a}^\dagger + \hat{a})/\sqrt{2}$. Such a term will replace the quantum gate given by the unitary operator U and therefore the evolution of the system is governed now by the oscillator algebra above and the system is controllable in the von Neumann sense for all initial states. The Jaynes–Cummings models provide many different realizations of such an interaction term [10, 16], as well as other possibilities for interaction terms that could be easily implemented in a concrete experimental setting.

5.2. Compact Lie groups

Once the connection between von Neumann controllability and the theory of unitary representations has been established, we can start a systematic discussion on the controllability of quantum systems by discussing the irreducible unitary representations of their dynamical Lie groups. Unfortunately, this programme cannot be fully carried on because there is not a complete theory of irreducible representations for all Lie groups. However, we have a fairly well-developed theory for various families of Lie groups. We will not pretend here to cover such a broad scenario and concentrate on two opposite cases which are both extremely useful, on physical and mathematical grounds, compact and nilpotent Lie groups. For both families of Lie groups their representations are very well known. The theory of representations of compact Lie groups was developed at the beginning of the 20th century by Weyl together with the creation of the mathematical foundations of quantum mechanics. The theory of unitary representations of nilpotent Lie groups took more time to mature, and Kirillov in 1960 proved the celebrated theorem establishing a one-to-one correspondence between unitary irreducible representations of nilpotent Lie groups and coadjoint orbits in the dual of the Lie algebra of the group, marking the beginning of the 'method of orbits' leading to the modern paradigm of geometric quantization.

The Peter–Weyl theorem establishes that the set $\hat{G}$ of equivalence classes of irreducible unitary representations of a compact Lie group G is a countable discrete set. Moreover that the regular representation of the group, i.e. the Hilbert space $L^2(G, \mu_G)$ where μ_G is the Haar measure of the group, decomposes as

$$L^2(G, \mu_G) = \bigoplus_{\alpha\in\hat{G}} \mathcal{H}_\alpha \otimes \mathcal{H}'_\alpha,$$

where the finite-dimensional $\mathcal{H}_\alpha$ is the support space of the irreducible unitary representation labelled by α and $\mathcal{H}'_\alpha$ is its dual space. If we denote by d_α the dimension of $\mathcal{H}_\alpha$, the previous formula also tells us that the multiplicity of the irreducible representation supported at $\mathcal{H}_\alpha$ on the regular representation is d_α. Hence, if we denote by R the regular representation, this is $(R(g)\psi)(g') = \psi(gg')$, $g, g' \in G$, $\psi \in L^2(G, \mu_G)$, we have

$$R = \bigoplus_{\alpha \in \hat{G}} d_\alpha U_\alpha,$$

where $U_\alpha: G \to U(\mathcal{H}_\alpha)$ denotes the corresponding irreducible unitary representation of G. Hence for any affine-linear quantum system defining a unitary representation of a compact Lie group G, the system will be von Neumann controllable iff $\mathcal{H} = \mathcal{H}_\alpha$ for some $\alpha \in \hat{G}$. Moreover, the system in general will not be controllable unless $\dim G \geqslant d_\alpha^2$. In fact, if the system is von Neumann controllable and hence it supports the irreducible representation $\mathcal{H}_\alpha$ then the map $U: G \to U(\mathcal{H}_\alpha)$ maps G as a subgroup of $U(\mathcal{H}_\alpha) \cong U(d_\alpha)$. Thus the system will be controllable if G is mapped surjectively on $U(d_\alpha)$ and this will only occur if $\dim G \geqslant d_\alpha^2$.

5.3. Nilpotent Lie groups

According to Kirillov's theorem any unitary irreducible representation of a nilpotent group G can be constructed by the induction of one-dimensional representations of suitable chosen subgroups H of G. Consider the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of the Lie group G. Let $\mu \in \mathfrak{g}^*$ and $\mathcal{O}_\mu$ be the coadjoint orbit through it, this is, $\mathcal{O}_\mu = \{\mathrm{Ad}_g^*\mu \mid g \in G\}$, where Ad_g^* denotes the adjoint of the adjoint action Ad_g of G on $\mathfrak{g}$. Let $H = G_\mu$ be the isotropy group of μ, i.e. $G_\mu = \{g \in G \mid \mathrm{Ad}_g^*\mu = \mu\}$ and $\mathfrak{h}$ its Lie algebra. Let us consider the character of H defined by μ, this is, $\chi_\mu(\exp\zeta) = \mathrm{e}^{\mathrm{i}\langle\mu,\zeta\rangle}$. The character χ_μ defines a one-dimensional unitary representation of the subgroup H of G. Then the unitary representation of G obtained by the induction from χ_μ, U^{χ_μ} is irreducible. Recall that U^{χ_μ} is an infinite-dimensional representation whose support space is the Hilbert space of square integrable functions $\psi: G \to \mathbb{C}$ such that $\psi(hg) = \chi_\mu(h)^*\psi(g)$, $g \in G, h \in H$; moreover, $U^{\chi_\mu}(g)\psi(g') = \psi(gg')$. According to this construction, if an affine-linear quantum control system defining a nilpotent dynamical Lie group is von Neumann controllable, i.e. it defines an irreducible representation of G, it is infinite dimensional; hence it will not be pure state controllable. Note that the only states reachable from a given one ψ_0 will be those translated by elements of the group, thus conforming a finite-dimensional orbit on state space. It is not obvious, however, whether the given system is going to be approximately controllable. Let us consider the simple example of a controlled free particle on $\mathbb{R}$:

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial q^2} - u(t)q\psi = (H_0 + uH_1)\psi. \tag{5.1}$$

Then the dynamical Lie algebra generated by H_0 and H_1 is a four-dimensional nilpotent Lie algebra that contains the Heisenberg algebra. The Hilbert space $L^2(\mathbb{R})$ supports an (essentially unique) unitary representation of the Heisenberg group according to von Neumann's theorem, thus it supports an irreducible unitary representation of the dynamical Lie group, hence the system will be von Neumann controllable. We cannot apply Chambrion's theorem to it because the spectrum of the operator H_0 is not discrete. However, it is easy to check that the controlled free particle above is not approximately controllable.

6. Conclusions and outlook

We have presented a new notion of control, von Neumann controllability, for quantum states that extends the usual notion of state controllability using the linear superposition principle of quantum mechanics. The output states are a linear superposition of states each obtained by evolving the input state using the control Hamiltonian with various control functions. A quantum system will be von Neumann controllable with respect to a given initial state if such a state is a cyclic vector for the unitary representation of the dynamical group generated by the control Hamiltonian, and the system will be said to be von Neumann controllable if it is controllable in this sense for all initial states. The notion of von Neumann controllability is equivalent to the irreducibility of such a unitary representation if and only if the system is von Neumann controllable with respect to all unitary vectors. The characterization of states such that the system is von Neumann controllable with respect to them for a given reducible representation of the dynamical group is an open question that will be addressed in forthcoming works.

We have also shown that the notion of von Neumann controllability is weaker than the notion of approximately pure state controllability because there are von Neumann controllable systems (those supporting irreducible representations of the dynamical group) that cannot be approximately controllable because the dimension of the representation space is much larger than the dimension of the group. On the other hand, von Neumann controllability can be applied without further difficulties both to finite- and infinite-dimensional systems. We have started the analysis of some simple infinite-dimensional systems which are von Neumann controllable by using some rudiments of the theory of irreducible unitary representations of nilpotent Lie groups. The problem becomes much more interesting when considering solvable groups and we leave it for further analysis.

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